REMARKS ON STEP COCYCLES OVER ROTATIONS, CENTRALIZERS AND COBOUNDARIES

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Abstract. By using a cocycle generated by the step function $\varphi_{\beta,\gamma} = 1_{[0,\beta]} - 1_{[0,\beta]}(. + \gamma)$ over an irrational rotation $x \to x + \alpha \bmod 1$, we present examples which illustrate different aspects of the general theory of cylinder maps. In particular, we construct non ergodic cocycles with ergodic compact quotients, cocycles generating an extension $T_{\alpha,\varphi}$ with a small centralizer. The constructions are related to diophantine properties of α, β, γ.

Contents

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Introduction

Skew maps (also called cylindrical systems) yield an important source of examples of dynamical systems preserving an infinite invariant measure. In particular the class of skew maps over 1-dimensional irrational rotations using a step function as skewing function has been widely studied in the literature. (cf. [23], [18], [3] and for other references [7]).

Our main examples here will be the cocycles generated over an irrational rotation $T_\alpha : x \to x + \alpha \bmod 1$ by the step functions[1]

$$\varphi_\beta(x) := 1_{[0,\beta]}(x) - \beta,\ \varphi_{\beta,\gamma}(x) := 1_{[0,\beta]}(x) - 1_{[0,\beta]}(x+\gamma).$$

This simple function can be used to answer natural questions about cocycles. In particular, we are interested in the construction of non ergodic cocycles with ergodic compact quotients and cocycles generating an extension $T_{\alpha,\varphi} : (x, y) \to (x + \alpha, y + \varphi(x))$ with a small centralizer. This has the advantage to illustrate the general ergodic theory of dynamical systems in infinite measure through a very elementary and natural object.

After reminders on extensions of dynamical systems, essential values and regularity of cocycles, we discuss some issues on $\mathbb{Z}^2$-cocycles and centralizer of cylindrical maps. Then we present general results on coboundaries equations over rotations and recall results of M. Guénais and F. Parreau on a multiplicative quasi-coboundary equation. In the case of step functions, we give sufficient conditions for solving in $L^2(\mathbb{T}^1)$ the linear coboundary equation for the function $1_{[0,\beta]} - T_\gamma 1_{[0,\beta]}$.

As a result, it follows (Theorem 3.1) that there are real numbers β such that:
- on one hand, for almost every γ the cocycle defined by $\varphi_{\beta,\gamma}$ is non regular (in particular it is not ergodic, but not a coboundary), but all the compact quotients of the associated skew product are ergodic,
- on the other hand, there is an uncountable set of values of γ for which $\varphi_{\beta,\gamma}$ is a coboundary.

Then we show different kinds of centralizer for T_{α,φ_β}: non trivial uncountable (case of unbounded partial quotient), trivial (case of bounded quotients). At the opposite we investigate also a property of "rigidity" for α of bounded type, with an example of cocycle φ which generates an extension $T_{\alpha,\varphi}$ with a small centralizer. A last application is the construction of a counter example in a conjugacy problem for a group family. In the appendix, under diophantine conditions on β, γ, we solve the linear coboundary equation for $\varphi_{\beta,\gamma}$.

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[1] In what follows the arguments of the functions are taken modulo 1.

1. **Preliminaries on cocycles**

1.1. **Cocycles and group extension of dynamical systems.**

In these preliminaries, we recall some standard facts on skew products and regular cocycles.

Let $(X, \mathcal{A}, \mu, T)$ be a dynamical system, i.e., a probability space $(X, \mathcal{A}, \mu)$ and a measurable invertible transformation T of X which preserves μ. In the sequel we will *assume T ergodic*. Let $\varphi : X \to G$ be a measurable function from X to an abelian locally compact second countable (lcsc) group G, with m or m_G denoting the Haar measure on G.

The *skew product (or cylinder map)* over (X, μ, T) with the fiber G and the displacement (or skewing) function φ is the dynamical system $(X \times G, \mu \otimes m, T_\varphi)$, where

$$T_\varphi(x, g) = (Tx, g + \varphi(x)).$$

For $n \in \mathbb{Z}$ we have $T_\varphi^n(x, g) = (T^n x, g + \varphi_n(x))$, where (φ_n) is the associated cocycle generated by φ over the dynamical system:

$$\varphi_n(x) = \sum_{j=0}^{n-1} \varphi(T^j x), \;\; n \geq 1. \tag{1}$$

For simplicity, the function φ itself will be called a *cocycle*. We say that a cocycle $\varphi : X \to G$ is ergodic if the transformation T_φ is ergodic on $X \times G$ for the measure $\mu \times m_G$.

Recall that two cocycles φ and ψ over a dynamical system (X, μ, T) are *cohomologous* with *transfer function* η, if there is a measurable map $\eta : X \to G$ such that[2]

$$\varphi = \psi + T\eta - \eta. \tag{2}$$

φ is a *μ-coboundary* if it is cohomologous to 0.

Recurrence: When G is non compact, to deal with extensions with a non dissipative behavior, it is desirable that a recurrence property holds. A point $x \in X$ is *recurrent* for the cocycle φ, if $\varphi_n(x) \not\to \infty$ when n tends to ∞. We say that φ is *recurrent* if a.e. $x \in X$ is recurrent. If the cocycle is recurrent, then the map T_φ is conservative for the invariant σ-finite measure $\mu \times m_G$.

An integrable cocycle φ with values in $\mathbb{R}$ is recurrent if and only if $\int \varphi \, d\mu = 0$ (cf. [24]). If φ is a recurrent cocycle, than every cocycle cohomologous to φ is recurrent.

[2]If f is a function defined on a space X and T a transformation on X, we write simply Tf for the composed function $f \circ T$. The equalities between functions are understood μ-a.e.

1.2. **Essential values, non regular cocycle.**

First we recall the notion of essential values of a cocycle (cf. K. Schmidt [24], see also J. Aaronson [1]).

Let φ be a cocycle with values in an abelian lcsc group G. If G is a non compact group, we add to G a point at ∞ with the natural notion of neighborhood.

Definition 1.1. An element $a \in G \cup \{\infty\}$ is an *essential value* of the cocycle φ (over the system (X, μ, T)) if, for every neighborhood $V(a)$ of a, for every measurable subset B of positive measure,

$$\mu\big(B \cap T^{-n}B \cap \{x : \varphi_n(x) \in V(a)\}\big) > 0, \text{ for some } n \in \mathbb{Z}. \tag{3}$$

We denote by $\overline{\mathcal{E}}(\varphi)$ the set of essential values of the cocycle φ and by $\mathcal{E}(\varphi) = \overline{\mathcal{E}}(\varphi) \cap G$ the *set of finite essential values.*

The set $\mathcal{E}(\varphi)$ is a closed subgroup of G, with $\mathcal{E}(\varphi) = G$ if and only if $(X \times G, \mu \otimes m, T_\varphi)$ is ergodic.

Two cohomologous cocycles have the same set of essential values. φ is a coboundary if and only if $\overline{\mathcal{E}}(\varphi) = \{0\}$.

Definition 1.2. We say that the cocycle defined by φ is *regular*, if φ can be reduced by cohomology to an ergodic cocycle ψ with values in the closed subgroup $\mathcal{E}(\varphi)$:

$$\psi = \varphi + \eta - T\eta, \tag{4}$$

Let us recall some of the properties of regular cocycles. A cocycle φ is regular if and only if $\varphi/\mathcal{E}(\varphi)$ is a coboundary. A regular cocycle is recurrent. In the regular case there is a "nice" ergodic decomposition of the measure $\mu \times m_G$ for the skew map T_φ: any T_φ-invariant function can be written $F(y - \eta(x))$ for a function F which is invariant by translation by elements of $\mathcal{E}(\varphi)$, with η given by (4).

If the cocycle is non regular, then the measures μ_x on X on which is based the ergodic decomposition of $\mu \otimes m$ are infinite, singular with respect to the measure μ and there are uncountably many of them pairwise mutually singular (cf. K. Schmidt [24], see also [8] for a complete description of the ergodic decomposition in the general case of non abelian lcsc groups G).

The following lemma is a simple tool which can be used to construct non regular cocycles.

Lemma 1.3. *If φ is a $\mathbb{Z}$-valued cocycle such that there exists $s \notin \mathbb{Q}$ for which the multiplicative coboundary equation $e^{2\pi i s\varphi} = \psi/T\psi$ has a measurable solution ψ, then $\mathcal{E}(\varphi) = \{0\}$. If φ is not a coboundary, then $\overline{\mathcal{E}}(\varphi) = \{0, \infty\}$ and the cocycle φ is non regular.*

Proof. From the hypothesis we have $\varphi = s^{-1}\zeta + \eta - T\eta$, where ζ has values in $\mathbb{Z}$. The cocycle φ can be viewed as a real cocycle with values in $\mathbb{Z}$, which is cohomologous to a cocycle with values in the closed subgroup $s^{-1}\mathbb{Z}$, with $s^{-1} \notin \mathbb{Q}$.

In general, if a cocycle φ is cohomologous to φ_1 and to φ_2, two functions with values respectively in closed subgroups with an intersection reduced to $\{0\}$, then $\mathcal{E}(\varphi) = \mathcal{E}(\varphi_1) \cap \mathcal{E}(\varphi_2) = \{0\}$. $\square$

Cocycles and ergodicity in compact quotients

If G is compact, then there exist a measurable function $\eta : X \to G$ such that, for the cocycle $\psi(x) = \varphi(x) + \eta(x) - \eta(Tx) \in \mathcal{E}(\varphi)$, the map T_ψ is ergodic on $X \times \mathcal{E}(\varphi)$. Therefore T_φ is regular.

Ergodicity implies ergodicity for all compact quotients $X \times G/G_0$, where G_0 is any cocompact closed subgroup of G. The converse does not hold in general.

A question is to find examples of skew products which are non ergodic on $X \times G$, but ergodic on all compact quotients $X \times G/G_0$.

There are example of skew products for which all compact quotients are ergodic. For instance, the directional billiard in the plane with periodic rectangular obstacles yields such examples: for almost every direction the compact quotients of the directional billiard are ergodic; nevertheless, due to recent results of K. Frączek and C. Ulcigrai ([10]), it is known that the billiard map is non ergodic and even non regular for a.e. parameters. This provides examples, but we would like to construct more elementary explicit examples (see Subsection 3.1).

Remark 1. Let φ be a cocycle with values in $G = \mathbb{R}^d \times \mathbb{Z}^{d'}$. If all of its compact quotients are ergodic, then φ is ergodic or non regular. Indeed, if φ is regular, then $\varphi/\mathcal{E}(\varphi)$ is a coboundary. Hence, if the compact quotients are ergodic for φ, the compact quotients of $G/\mathcal{E}(\varphi)$ are trivial. This implies $\mathcal{E}(\varphi) = G$ and φ is ergodic.

1.3. $\mathbb{Z}^2$-actions and centralizer.

1.3.1. $\mathbb{Z}^2$-*actions and skew maps.*

The construction of skew maps can be extended to group actions generalizing the action of $\mathbb{Z}$ generated by iteration of a single automorphism. We consider the case of $\mathbb{Z}^2$-actions.

Let T_1, T_2 be two commuting measure preserving invertible transformations on (X, μ). They define a $\mathbb{Z}^2$-action on (X, μ). A G-valued function $\varphi(n_1, n_2, x)$ on $\mathbb{Z}^2 \times X$ is a cocycle for this action, if it satisfies the cocycle relation:

$$\varphi(n_1 + n_1', n_2 + n_2', x) = \varphi(n_1, n_2, x) + \varphi(n_1', n_2', T_1^{n_1} T_2^{n_2} x), \forall\, n_1, n_1', n_2, n_2' \in \mathbb{Z}.$$

Let φ_i, $i = 1, 2$, be two measurable G-valued functions on X and consider the skew products $\tilde{T}_i : (x, y) \longrightarrow (T_i x, y + \varphi_i(x))$ on $X \times \mathbb{R}$. Do they generate a $\mathbb{Z}^2$-action which extends the $\mathbb{Z}^2$-action on (X, μ)?

The maps $\tilde{T}_1$ and $\tilde{T}_2$ commute if and only if the following coboundary equation is satisfied

$$\varphi_1 - T_2\varphi_1 = \varphi_2 - T_1\varphi_2. \tag{5}$$

If (5) is satisfied, then the composed transformation $\tilde{T}_2^{n_2}\tilde{T}_1^{n_1}$ reads:

$$\tilde{T}_2^{n_2}\tilde{T}_1^{n_1}(x,y) = (T_2^{n_2}T_1^{n_1}x, y + \varphi(n_1,n_2,x)),$$

where $\varphi(1,0,x) = \varphi_1(x)$, $\varphi(0,1,x) = \varphi_2(x)$, $\varphi(n_1,n_2,x)$ satisfies the cocycle relation and $(n_1,n_2) \to \tilde{T}_2^{n_2}\tilde{T}_1^{n_1}$ defines a measure preserving action of $\mathbb{Z}^2$ on $X \times G$.

Therefore it is equivalent to find G-valued $\mathbb{Z}^2$-cocycles (and the corresponding skew products) or to find pairs (φ_1,φ_2) satisfying (5).

Clearly if $\varphi_1 = v - T_1\, v$ for some measurable function v, then Equation (5) holds with $\varphi_2 = v - T_2\, v$. A question is the construction of a pair (φ_1,φ_2) which satisfies (5) but are not of this form. In other words, can we *construct solutions of (5) which are not coboundaries?*

The answer depends on the choice of the transformations and on the class to which the functions φ_1, φ_2 belong. For instance, there is a "rigidity" for the $\mathbb{Z}^2$-shift on $\{0,1\}^{\mathbb{Z}^2}$ endowed with the product measure. When the functions are locally constant, the only solutions in that case are the trivial ones (cf. K. Schmidt [25], O. Jenkinson [14]).

In the case of rotations on $\mathbb{T}^1$, by using Fourier analysis, we will give below explicit examples of non degenerate solutions of (5) in $L^2(\mathbb{T}^1)$ (Theorem 2.2) and apply it to the construction of non trivial centralizers, a notion that we recall now.

1.3.2. *Centralizer of the cylinder product.*

A problem related to the construction of $\mathbb{Z}^2$-cocycles is the study of the centralizer.

In what follows[3] by *centralizer* of a cylinder map $\tilde{T}_1 : (x,y) \to (T_1x, y + \varphi_1)$, we mean the group $\mathcal{C}(\tilde{T}_1)$ of *measure preserving automorphisms* of $(X \times G, \mu \times m_G)$ which commute with $\tilde{T}_1$. It contains the powers of the map and the translations on the fibers. The skew products of the form $(x,y) \to (T_2x, y+\varphi_2)$ with T_2 commuting with T_1 and (φ_1,φ_2) satisfying (5) are elements of the group $\mathcal{C}(\tilde{T}_1)$.

1.4. **Case of an irrational rotation.**

In this subsection, we take the dynamical system (X,μ,T) in the class of rotations on $\mathbb{T}^1$ (which could be replaced by a compact abelian group K). For simplicity we take cocycles with values in $\mathbb{R}$. About the centralizer and related questions, see [15] (for G a compact group), [16], [2], [3], [17].

[3]The centralizer, in a wider sense, is the collection of non-singular transformations of X which commute with $\tilde{T}_1$ (see for instance [3]).

In the sequel α will be an irrational number and T_α the ergodic rotation $x \to x+\alpha \bmod 1$ on $X = \mathbb{T}^1$. For a measurable function $\varphi : X \to \mathbb{R}$, we consider the skew product $T_{\alpha,\varphi} : (x,y) \to (x+\alpha, y+\varphi(x))$.

In this case, according (5), the automorphisms given by skew products of the form $T_{\gamma,\psi} : (x,y) \to (x+\gamma, y+\psi)$, for $\gamma \in \mathbb{T}^1$ and a measurable function ψ with (φ,ψ) satisfying $\varphi - T_\gamma\varphi = \psi - T_\alpha\psi$ are elements of the group $\mathcal{C}(T_{\alpha,\varphi})$. A problem is to find all elements in $\mathcal{C}(T_{\alpha,\varphi})$.

The following result is a special case of Proposition 1.1 in [2].

Theorem 1.4. *(cf.* [2]*) Suppose that the cocycle generated by φ over the rotation T_α is ergodic. Then any automorphism of $(X \times \mathbb{R}, \mu \times dy)$ commuting with $T_{\alpha,\varphi}$ has the form $(x,y) \to (x+\gamma, \varepsilon y + \psi(x))$ where $\gamma \in \mathbb{T}^1$, ε is a constant in ± 1 and $\psi : X \to \mathbb{R}$ is a measurable function such that*

$$\varepsilon\varphi - T_\gamma\varphi = \psi - T_\alpha\psi. \tag{6}$$

Proof. We give a sketch of the proof. The measure theoretic details are omitted. Let $\tilde{T}_2$ be an automorphism which commutes with $\tilde{T}_1 := T_{\alpha,\varphi}$. With the notation $u(x,y) = e^{2\pi i x}$, we deduce from the commutation $\tilde{T}_1\tilde{T}_2 = \tilde{T}_2\tilde{T}_1$, that $\tilde{T}_2 u$ is an eigenfunction for $\tilde{T}_1$ with eigenvalue $e^{2\pi i\alpha}$. By ergodicity of $\tilde{T}_1$, this implies that $u \circ \tilde{T}_2 = \lambda u$, where λ is a complex number of modulus 1.

It follows that $\tilde{T}_2$ leaves invariant the rotation factor of $\tilde{T}_1$ and that there are $\gamma \in \mathbb{R}$ and a measurable map $(x,y) \to V(x,y)$ from $X \times \mathbb{R}$ to $\mathbb{R}$ such that $\tilde{T}_2$ can be represented as $(x,y) \to \tilde{T}_2(x,y) = (x+\gamma, V(x,y))$.

The commutation of the maps $\tilde{T}_1, \tilde{T}_2$ implies:

$$V(x+\alpha, y+\varphi(x)) = V(x,y) + \varphi(x+\gamma). \tag{7}$$

Let us define $u_z(x,y) := V(x,y) - V(x,y+z)$, for $x \in X, y, z \in \mathbb{R}$. Using (7), we obtain:

$$u_z(x+\alpha, y+\varphi(x)) = V(x+\alpha, y+\varphi(x)) - V(x+\alpha, y+z+\varphi(x))$$
$$= V(x,y) + \varphi(x+\gamma) - [V(x,y+z) + \varphi(x+\gamma)] = V(x,y) - V(x,y+z) = u_z(x,y).$$

Therefore u_z is $\tilde{T}_1$-invariant, hence, by ergodicity of $T_{\alpha,\varphi}$, for every z, $u_z(x,y)$ is a.e. equal to a constant $c(z)$.

Since u_z satisfies $u_{z_1+z_2}(x,y) = u_{z_1}(x,y) + u_{z_2}(x,y+z_1)$, the previous relation implies $c(z_1+z_2) = c(z_1) + c(z_2)$; hence, since c is measurable, $c(z) = \lambda z$ for a constant λ.

So we have for every z, for a.e. (x,y) the relation $V(x,y+z) = V(x,y) - \lambda z$. By Fubini it follows that for a.e. y, for a.e. (x,z): $V(x,y+z) = V(x,y) - \lambda z$.

Therefore, for some $y_1 \in \mathbb{R}$ we have $V(x,z+y_1) = V(x,y_1) - \lambda z$; hence, setting $\psi(x) = V(x,y_1) + \lambda y_1$, we obtain for a.e. (x,z): $V(x,z) = \psi(x) - \lambda z$. Since the Lebesgue measure is preserved on $\mathbb{R}$ by the map $\tilde{T}_2$, necessarily $\lambda = +1$ or $\lambda = -1$.

Finally, the transformation $\tilde{T}_2$ has the form $(x,y) \to (x+\gamma, y+\psi(x))$ or $(x,y) \to (x+\gamma, -y+\psi(x))$. $\square$

Remark that the analogous result with $\mathbb{R}$ replaced by $\mathbb{Z}$ holds for a cocycle with values in $\mathbb{Z}$ which is ergodic for the action on $X \times \mathbb{Z}$.

Groups associated to a cocycle

>From Equation (6) it follows that $\varphi - T_{2\gamma}\varphi$ is coboundary:

$$\varphi - T_{2\gamma}\varphi = (\varepsilon\psi + T_\gamma\psi) - T_\alpha(\varepsilon\psi + T_\gamma\psi). \tag{8}$$

Now we define several groups related to the centralizer of $T_{\alpha,\varphi}$:

$$\Gamma := \{\gamma : \text{ for } \varepsilon = +1 \text{ or } \varepsilon = -1, \varepsilon\varphi - T_\gamma\varphi \text{ is a coboundary for } T_\alpha\},$$
$$\Gamma_0 := \{\gamma : \varphi - T_\gamma\varphi \text{ is a coboundary } \psi_\gamma - T_\alpha\psi_\gamma \text{ for } T_\alpha\}.$$

By (8) we have $2\Gamma \subset \Gamma_0 \subset \Gamma$. For $\gamma \in \Gamma_0$, ψ_γ is unique up to a constant. The family $\{\psi_\gamma, \gamma \in \Gamma_0\}$ satisfies the cocycle property on $\Gamma_0 \times X$ (up to a constant).

For $p \in [1, \infty]$ we define

$$\Gamma_p := \{\gamma \in \Gamma_0 : \psi_\gamma \in L^p(\mu)\}, \quad \mathcal{C}_p(T_{\alpha,\varphi}) = \{T_{\gamma,\psi_\gamma}, \gamma \in \Gamma_p\}. \tag{9}$$

If $\gamma \in \Gamma_1$, we can choose ψ_γ with zero mean. The group $\mathcal{C}_1(T_{\alpha,\varphi})$ is abelian. The cocycle property is satisfied by $\{\psi_\gamma, \gamma \in \Gamma_1\}$: for every γ, γ' in Γ_1, we have the relation:

$$\psi_{\gamma'+\gamma} = \psi_\gamma + \psi_{\gamma'}(. + \gamma) = \psi_{\gamma'} + \psi(. + \gamma').$$

A general result on coboundaries for rotations

Now we show that Γ_0 is a small group unless the cocycle φ is a coboundary, which is the degenerate case.

Let us consider the general case of rotations on a compact abelian group K. For $\gamma \in K$, T_γ denotes the rotation (translation) by γ on K. Let T_α be a given ergodic rotation on K defined by an element $\alpha \in K$.

The following proposition is an easy consequence of Theorem 6.2 in [21] and of the proposition p. 178 in ([19]) (Lemma 1.6 below).

Proposition 1.5. *Let φ be a measurable function on K. If for every γ in a set of positive measure in K there exists a measurable function ψ_γ such that $\varphi - T_\gamma\varphi = \psi_\gamma - T_\alpha\psi_\gamma$, then φ is an additive quasi-coboundary:*

$$\varphi = c + T_\alpha h - h,$$

for a measurable function h and a constant c. If φ is integrable, then $c = \int \varphi \ \mu$.

Lemma 1.6. *([19]) Let φ be a measurable real function on K. If $e^{2\pi i(\varphi - T_\gamma \varphi)}$ is a T_α-coboundary for every γ in a subset of positive measure in K, then there are a measurable function ζ_s of modulus 1 and λ_s of modulus 1 such that: $e^{2\pi i s\varphi} = \lambda_s \, T_\alpha \zeta_s / \zeta_s$.*

2. Coboundary equations for irrational rotations

This section is devoted to the coboundary equations over irrational rotations, either linear equations (with Fourier's series methods) or multiplicative equation (Guenais-Parreau's results). The following step functions are used:

Notation Let β be a fixed real number. For any real number γ, with the notation T_γ for the translation $x \to x + \gamma$ mod 1, we will consider the cocycles generated over an irrational rotation T_α by the step functions

$$\varphi_\beta = 1_{[0,\beta]} - \beta, \;\; \varphi_{\beta,\gamma} := 1_{[0,\beta]} - 1_{[0,\beta]}(. + \gamma) = \varphi_\beta - T_\gamma \varphi_\beta. \tag{10}$$

2.1. Classical results, expansion in basis $q_n\alpha$.

First of all we recall classical facts on continued fractions and on expansion of a real β in basis "$q_n\alpha$" (Ostrowski expansion).

In the following $\alpha \in]0, 1[$ is an irrational number and $[0; a_1, ..., a_n, ...]$ is its continued fraction expansion. Let $(p_n/q_n)_{n\geq 0}$ be the sequence of its convergents. Recall that $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$ and, for $n \geq 1$:

$$p_n = a_n p_{n-1} + p_{n-2}, \;\; q_n = a_n q_{n-1} + q_{n-2}, \;\; (-1)^n = p_{n-1}q_n - p_n q_{n-1}. \tag{11}$$

Notations For $u \in \mathbb{R}$, put $[u]$ for its integral part and $\|u\| := \inf_{n\in\mathbb{Z}} |u - n|$.

For $n \geq 0$ we have $\|q_n\alpha\| = (-1)^n(q_n\alpha - p_n)$ and the following inequalities (cf. [12]):

$$\begin{aligned} 1 &= q_n\|q_{n+1}\alpha\| + q_{n+1}\|q_n\alpha\|, \\ \frac{1}{2}\frac{1}{q_{n+1}} \leq \frac{1}{q_{n+1}+q_n} &\leq \|q_n\alpha\| \leq \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}}, \qquad (12)\\ \frac{1}{2}\frac{1}{q_{n+1}} \leq \|q_n\alpha\| &\leq \|k\alpha\|, \text{ for } 1 \leq k < q_{n+1}. \qquad (13) \end{aligned}$$

An irrational number $\alpha = [0; a_1, ..., a_n, ...]$ has *bounded partial quotients* (abbreviated in "*is of bounded type*") if the sequence (a_n) is bounded.

Expansion in basis $q_n\alpha$ (Ostrowski expansion)

For $\beta \in \mathbb{T}^1$ we consider the following representation introduced by Ostrowski (1921)

$$\beta = \sum_1^\infty b_j(\beta)\, q_j\, \alpha \text{ mod } 1, \tag{14}$$

where $(b_j(\beta))_{j\geq 0}$ is a sequence in $\mathbb{Z}$.

Any $\beta \in \mathbb{T}^1$ has such an expansion. If $\sum_{j\geq 1} \frac{|b_j(\beta)|}{a_{j+1}} < \infty$, the representation is unique up to a finite number of terms. It is shown in [11] that this condition is equivalent to $\sum \|q_j\beta\| < \infty$. For $r \geq 1$, we call $H_r(\alpha)$ the subgroup

$$H_r(\alpha) := \{\beta = \sum_1^\infty b_j(\beta)\, q_j\, \alpha \text{ mod } 1 : \sum_{j\geq 1} \frac{|b_j(\beta)|^r}{a_{j+1}} < \infty\}.$$

2.2. **Linear and multiplicative equations for φ_β and $\varphi_{\beta,\gamma}$.**

2.2.1. *Fourier conditions.*

For $\varphi(x) = \sum_{n\in\mathbb{Z}} \varphi_n e^{2\pi i n x}$ in $L^1(\mathbb{T})$ with $\int \varphi\, d\mu = 0$, if the coboundary equation $\varphi = h - T_\alpha h$ has a solution $h \in L^1(\mathbb{T})$, the Fourier coefficients of h are $h_n = \frac{\varphi_n}{1-e^{2\pi i n\alpha}}$. Therefore the necessary and sufficient condition for the existence of a L^2 solution, for $\varphi \in L^2(\mathbb{T})$ is $\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{|\varphi_n|^2}{\|n\alpha\|^2} < +\infty$.

As it is well known, under diophantine assumptions on α and regularity of the function φ, the coboundary equations can be solved. We recall briefly this fact.

The type of an irrational number α is $\eta \geq 1$ such that

$$\inf_{k\neq 0} [k^{\eta-\varepsilon}\|k\alpha\|] = 0, \quad \inf_{k\neq 0}[k^{\eta+\varepsilon}\|k\alpha\|] > 0, \ \forall\varepsilon > 0. \tag{15}$$

Recall that the type of a.e. α is 1. From a result of V. I. Arnold ([4]) (see also M. Herman ([13])), we have:

Theorem 2.1. *([4]) If α is of type η and $\varphi(x) = \sum_{n\neq 0} \varphi_n e^{2\pi i n x}$ with $\varphi_n = O(n^{-(\eta+\delta)})$ and $\delta > 0$, then $\varphi_\gamma(x) := \sum_{n\neq 0} \varphi_n \frac{1-e^{2\pi i n\gamma}}{1-e^{2\pi i n\alpha}} e^{2\pi i n x}$ is a well defined continuous function for every γ and the pairs (α, φ), (γ, φ_γ) define commuting skew products.*

Clearly this is a degenerate coboundary case in accordance with Proposition 1.5, since we have a solution for every γ. This is a motivation to consider step functions like the function $\varphi_{\beta,\gamma}$ introduced above.

2.2.2. *The linear coboundary equation, a sufficient condition for $\varphi_{\beta,\gamma}$.*

Now we give sufficient conditions in case of the step function $\varphi_{\beta,\gamma}$ for the existence of a solution of the linear coboundary equation (5).

Recall that the cocycle φ_β is not a coboundary for $\beta \notin \mathbb{Z}\alpha + \mathbb{Z}$. This follows from the fact that $e^{2\pi i\varphi_\beta} = e^{-2\pi i\beta}$, hence $e^{-2\pi i\beta}$ is an eigenvalue of the rotation by if φ_β is a linear coboundary (cf. [22]). A stronger result is that the cocycle defined by φ_β over the rotation T_α is ergodic if $\beta \notin \mathbb{Q}\alpha + \mathbb{Q}$ (cf. Oren [23]).

The Fourier coefficients of $\varphi_{\beta,\gamma} = 1_{[0,\beta]} - 1_{[\gamma,\beta+\gamma]}$ are $\frac{1}{2\pi i n}(e^{2\pi i n\beta} - 1)(e^{2\pi i n\gamma} - 1)$. The condition for $\varphi_{\beta,\gamma}$ to be a coboundary with a transfer function in $L^2(\mathbb{T}^1)$, i.e., such

that the functional equation $\varphi_{\beta,\gamma} = T_\alpha h - h$ has a solution h in L^2, is

$$\sum_{n\neq 0} \frac{1}{n^2} \frac{\|n\beta\|^2 \|n\gamma\|^2}{\|n\alpha\|^2} < \infty. \tag{16}$$

For the cocycle $\varphi_{\beta,\gamma}$ the following result is proved in Appendix:

Theorem 2.2. *If $\beta \in \mathbb{T}^1$ is in $H_4(\alpha)$ then we have*

$$\sum_{n\neq 0} \frac{1}{n^2} \frac{\|n\beta\|^4}{\|n\alpha\|^2} < \infty.$$

If β, γ are in $H_4(\alpha)$, then (16) holds and there is $\psi_{\beta,\gamma}$ in $L^2(\mathbb{T}^1)$ solution of

$$1_{[0,\beta]} - T_\gamma 1_{[0,\beta]} = \psi_{\beta,\gamma} - T_\alpha \psi_{\beta,\gamma}. \tag{17}$$

Therefore, if α is not of bounded type (i.e., has unbounded partial quotients), there is an uncountable set of pairs of real numbers β and γ such that $\varphi_{\beta,\gamma} := 1_{[0,\beta]} - 1_{[0,\beta]}(. + \gamma)$ is a coboundary $\psi - T_\alpha \psi$ for T_α with ψ in L^2.

Remark that by Shapiro's result (cf. [26]) on the difference of two indicators of intervals, ψ is not in L^∞, unless β and γ are in $\mathbb{Z}\alpha + \mathbb{Z}$.

2.2.3. *Multiplicative equation: a necessary and sufficient condition.*

Now we consider the multiplicative functional equation for φ_β:

$$e^{2i\pi s \varphi_\beta} = e^{2i\pi t} \frac{T_\alpha f}{f}, \tag{18}$$

f is a measurable function which can be assumed of modulus 1.

Equation (18) was studied by W. Veech [27], then by K. Merril [20] who gave a sufficient condition on (β, s, t) for the existence of a solution. M. Guénais and F. Parreau have shown that this condition is sufficient and they have extended it to more general step functions:

Theorem 2.3. *(*[11]*, Theorems 1 and 2)*
a) Equation (18) has a measurable solution f for the parameters (β, s, t) if and only if there is a sequence of integers (b_n) such that:

$$\beta = \sum_{n\geq 0} b_n q_n \alpha \ mod\ 1, \ with \sum_{n\geq 0} \frac{|b_n|}{a_{n+1}} < \infty, \ \sum_{n \geq 0} \|b_n s\|^2 < \infty,$$
$$t = k\alpha - \sum_{n\geq 0} [b_n s]\, q_n \alpha \ mod\ 1, \ for\ an\ integer\ k.$$

b) Let $\varphi : \mathbb{T}^1 \to \mathbb{R}$ be a step function with integral 0 and jumps $-s_j$ at distinct points $(\beta_j, 0 \leq j \leq m)$, $m \geq 1$, and let $t \in \mathbb{T}$. Suppose that there is a partition $\mathcal{P}$ of $\{0, .., m\}$ such that for every $J \in \mathcal{P}$ and $\beta_J \in \{\beta_j, j \in J\}$ the following conditions are satisfied:

(i) $\sum_{j\in J} s_j \in \mathbb{Z}$;
(ii) for every $j \in J$, *there is a sequence of integers* $(b_n^j)_n$ *such that*

$$\beta_j = \beta_J + \sum_{n\geq 0} b_n^j q_n \alpha \ mod\ 1, \ with \ \sum_{n\geq 0} \frac{|b_n^j|}{a_{n+1}} < +\infty, \ \sum_{n\geq 0} \Big\| \sum_{j\in J} b_n^j s_j \Big\|^2 < +\infty;$$

(iii) $t = k\alpha - \sum_{J\in\mathcal{P}} t_J$, *with* $k \in \mathbb{Z}$ *and*

$$t_J = \beta_J \sum_{j\in J} s_j + \sum_{n\geq 0} \Big[\sum_{j\in J} b_n^j s_j\Big] q_n \alpha \ mod\ 1.$$

Then there is a measurable function f *of modulus 1 solution of*

$$e^{2i\pi\varphi} = e^{2i\pi t}\, T_\alpha f / f. \tag{19}$$

Conversely, when $\sum_{j\in J} s_j \notin \mathbb{Z}$ *for every proper non empty subset* J *of* $\{0, .., m\}$, *these conditions are necessary for the existence of a measurable solution of (19).*

Remark 2. In the situation of Theorem 2.2, the multiplicative equation for $s\varphi_{\beta,\gamma}$ has a solution for every $s \in \mathbb{R}$. Observe that the necessary condition of Theorem 2.3 b) does not apply to $s\varphi_{\beta,\gamma}$ (no condition on s). Indeed the set of discontinuities of $\varphi_{\beta,\gamma}$ is $J = \{0, \beta, -\gamma, \beta - \gamma\}$ with respective jumps: $+1, -1, +1, -1$. There is a decomposition of J into $J_1 = \{0, \beta\}$, $J_2 = \{-\gamma, \beta - \gamma\}$ and the sum of jumps is 0 for each of these subsets.

3. **Applications**

3.1. **Non ergodic cocycles with ergodic compact quotients.**

A first application of the results of Section 2 is the construction of simple examples of non regular cocycles with ergodicity of all compact quotients.

By using the sufficient condition of Theorem 2.3 a), we construct non regular (hence non ergodic) $\mathbb{Z}$-valued cocycles given by the step cocycles $\varphi_{\beta,\gamma}$ defined in (10) over rotations such that all compact quotients in $X \times \mathbb{Z}/a\mathbb{Z}$ are ergodic (see also [7], [9]).

Let us recall that for every irrational number α, for almost every (β, γ) the cocycle $\varphi_{\beta,\gamma}$ is ergodic[4]. Therefore clearly we are interested here in special, non generic, sets of values of (β, γ).

Theorem 3.1. *If* α *is not of bounded type, there is* β *in* $H_1(\alpha)$ *such that for a.e.* γ*:*
a) the cocycle $\varphi_{\beta,\gamma}$ *is non regular;*
b) all compact quotients $T_{\alpha,\varphi_{\beta,\gamma}}$ *mod* $a : (x, y \text{ mod } a) \to (x + \alpha, y + \varphi_{\beta,\gamma}(x) \text{ mod } a)$ *are ergodic.*

[4]See Théorème 5 in [5], where ergodicity is proved for $T_{\alpha,\varphi}$,when φ is a step function, under a generic condition on the discontinuity points of φ called Condition (A').

Proof. a) If α is not of bounded type, by Theorem 2.3 a) there is a non-countable set of values of β such that, for a non-countable set of values of s, there are a number λ of modulus 1 and a measurable function f of modulus 1 such that $e^{2\pi i s\varphi_\beta} = \lambda \frac{T_\alpha f}{f}$.

We can take $\beta \notin \alpha\mathbb{Z} + \mathbb{Z}$ and $s \notin \mathbb{Q}$. For this choice of β and of s, $e^{2\pi i s(\varphi_\beta - T_\gamma \varphi_\beta)}$ is a multiplicative coboundary for every γ.

On the other hand, if $1_{[0,\beta]} - T_\gamma 1_{[0,\beta]}$ is an additive coboundary for every γ in a set of positive measure, then by Proposition 1.5 this implies that $1_{[0,\beta]} - \beta$ is an additive coboundary which is not the case (cf. 2.2.2).

Therefore for a.e. $\gamma \in \mathbb{R}$, $\varphi_{\beta,\gamma}$ is not an additive coboundary. For such a value of γ, Lemma 1.3 shows that $\overline{\mathcal{E}}(\varphi_{\beta,\gamma}) = \{0, \infty\}$ and $\varphi_{\beta,\gamma}$ is non regular.

b) Now we construct in $H_1(\alpha)$ a more restricted set of β such that, for a.e. γ, the action of $T_{\alpha,\varphi_{\beta,\gamma}}$ on the compact quotients $X \times \mathbb{Z}/a\mathbb{Z}$ are ergodic for all $a \in \mathbb{Z} - \{0\}$.

This done is two steps: if α is of non bounded type, we construct $\beta \in H_1(\alpha)$ such that

$$\{s : \sum_n \|b_n(\beta)s\|^2 < \infty\} \cap \mathbb{Q} = \mathbb{Z}, \tag{20}$$

then show that this implies the desired property

1) There exists a strictly increasing sequence of integers (j_n) and a sequence of integers $(d_n \geq 1)$ such that, if one defines the subsequence (b_{j_n}) by

$$b_0 = 1,\ b_{-1} = 0,\ b_{j_{n+1}} = d_n b_{j_n} + b_{j_{n-1}} \text{ for } n \geq 1, \tag{21}$$

then the conditions $\sum(\frac{b_{j_n}}{a_{j_n+1}}) < \infty$ and $\sum(\frac{b_{j_n}}{b_{j_{n+1}}})^2 < \infty$ are satisfied. We complete the sequence (b_n) by zeroes.

For instance, we can choose $d_n = n$ for all $n \geq 1$ and then (j_n) such that the series $\sum \frac{n!}{a_{j_n+1}}$ converges.

The condition $\sum(\frac{b_{j_n}}{b_{j_{n+1}}})^2 < \infty$ insures the existence of an uncountable set of values of s such that $\sum_n \|b_n s\|^2 < \infty$. In particular, there is $s \notin \mathbb{Q}$ for which this condition holds.

Suppose that $\frac{u}{v}$, with u, v coprime integers, satisfies $\sum_n \|b_n \frac{u}{v}\|^2 < \infty$. For n big enough, v divides ub_n. As b_{j_n} and $b_{j_{n+1}}$ are mutually coprime (by the choice of initial values and Equation (21)), we have $v = \pm 1$.

2) Let β such that $b_n(\beta) = b_n$. We have shown above that (20) holds and the non regularity of $\varphi_{\beta,\gamma}$ for almost all γ. Now we prove that, for a.e. γ, all compact quotients of $T_{\alpha,\varphi_{\beta,\gamma}}$ are ergodic.

Let us suppose on the contrary that there is a set D of positive measure such that, for every $\gamma \in D$, there is an integer a such that $T_{\alpha,\varphi_{\beta,\gamma}}$ mod a is non ergodic.

Using Fourier series representation of $T_{\alpha,\varphi_{\beta,\gamma}}$-invariant a-periodic functions, this would imply the following: there are integers a and k, with a, k coprime, and a set $D_{a,k}$ of positive measure such that for every $\gamma \in D_{a,k}$ there exists a measurable function f_γ satisfying:

$$e^{-2i\pi \frac{k}{a}(\varphi_\beta - T_\gamma \varphi_\beta)(x)} = f_\gamma(x)/f_\gamma(x+\alpha). \tag{22}$$

Lemma 1.6 implies the existence of t and h such that

$$e^{-2i\pi \frac{k}{a}\varphi_\beta(x)} = e^{2i\pi t}\, h(x)/h(x+\alpha).$$

As the conditions in Theorem 2.3 a) are necessary, this implies that $\sum \|b_n \frac{k}{a}\|^2 < \infty$, contrary to (20).

Remark that, by strengthening the conditions in the construction of β, we can also find $\beta \in H_4(\alpha)$ with the previous properties. For such a β, by Theorem 2.2 there is an uncountable set of values of γ for which $\varphi_{\beta,\gamma}$ is a coboundary. $\square$

3.2. **Examples of non trivial and trivial centralizer.**

The results of Subsection 1.4 lead to the following questions for a given rotation T_α and a function φ:
- for which $\gamma \in \mathbb{T}^1$ is there a solution to the commutation equation $\varphi - T_\gamma \varphi = \psi - T_\alpha \psi$?
- what is the centralizer of $T_{\alpha,\varphi}$?

In this subsection, from Theorem 2.2 we obtain that the centralizer of T_{α,φ_β} is non countable for $\beta \in H_4(\alpha)$. Then we show that the centralizer $\mathcal{C}(T_{\alpha,\varphi_\beta})$ is also non trivial when $\beta \in H_1(\alpha)$. In a second part, we investigate a property of "rigidity" for α of bounded type, with an example of a small centralizer.

3.2.1. *Case of a non trivial centralizer.*

Let α be an irrational number which is not of bounded type and β a real number. Let us consider $\varphi = \varphi_\beta = 1_{[0,\beta]} - \beta$.

If T_{α,φ_β} is ergodic, by Theorem 1.4 and the commutation relation (8), the square of the elements of $\mathcal{C}(T_{\alpha,\varphi_\beta})$ are of the form $T_{\gamma,\psi}$ with ψ a measurable function and γ such that $1_{[0,\beta]}(.) - 1_{[0,\beta]}(. + 2\gamma) = \psi - T_\alpha \psi$.

By Theorem 2.2, if β is in $H_4(\alpha)$, the group Γ_2 defined in Subsection 1.3.2 contains the group $H_4(\alpha)$, which is a non countable group if α is not of bounded type.

Now we would like to weaken the condition on β and still get a non trivial centralizer. It is interesting to investigate the properties of the cocycle $\varphi_{\beta,\beta}$ or more generally $\varphi = a1_{[0,\beta]} - 1_{[0,a\beta]}$ with a a positive integer. This is a special situation where one can conclude that the cocycle is a coboundary by using the result of Guénais and Parreau mentioned above.

Proposition 3.2. *If a is a positive integer, the cocycle $\varphi = a1_{[0,\beta]} - 1_{[0,a\beta]}$ is a coboundary if and only if β is in $H_1(\alpha)$.*

Proof. With the notation of Theorem 2.3, the discontinuities of $\varphi = a1_{[0,\beta]} - 1_{[0,a\beta]}$ are at $\beta_0 = 0, \beta_1 = \beta, \beta_2 = \gamma = a\beta$, with jumps respectively $a-1, -a, 1$, we have $m = 2$ and the partition $\mathcal{P}$ is the trivial partition with the single atom $J = \{0, 1, 2\}$. We have $\beta_J = 0$, $\sum_{j\in J} s_j = 0$.

Suppose that $\beta \in H_1(\alpha)$ with an expansion in basis $(q_n\alpha)$ given by

$$\beta = \sum_{n\geq 0} b_n q_n \alpha \text{ mod } 1, \text{ with } \sum_{n\geq 0} \frac{|b_n|}{a_{n+1}} < +\infty, \ b_n \in \mathbb{Z}. \tag{23}$$

We can take $b_n^0 = 0, b_n^1 = b_n, b_n^2 = ab_n$, so that $\sum_{j\in J} b_n^j s_j = ab_n - ab_n = 0$. For every real s the multiplicative equation $e^{2\pi is\varphi} = T_\alpha f/f$ has a solution. By using Theorem 6.2 in [21], we conclude that φ is a measurable coboundary (another proof based on the tightness of the cocycle (that is, the tightness of the family $(\varphi_n, n \geq 0)$) can also be given).

Conversely, if φ is a measurable coboundary, then $e^{2\pi is\varphi} = T_\alpha f/f$ has a solution for every real s, and this implies that β has an expansion like in (23) (Theorem 2.3 b), necessary condition). $\square$

Under the assumption $\beta \in H_1(\alpha)$ which is weaker than the assumption of Theorem 2.2, Proposition 3.2 implies:

Corollary 3.3. *If $\beta \in H_1(\alpha)$, the centralizer $\mathcal{C}(T_{\alpha,\varphi_\beta})$ contains a non trivial element T_{β,ψ_β}, where ψ_β is a measurable function solution of $1_{[0,\beta]} - 1_{[\beta,2\beta]} = \psi_\beta - T_\alpha\psi_\beta$.*

Remark 3. We have seen in the previous considerations that, under some assumption on the expansion of β in basis $q_n\alpha$, the cocycle $\varphi_{\beta,\beta} = 1_{[0,\beta]} - T_\beta 1_{[0,\beta]}$ is a coboundary for the rotation by α, with a transfer function in a certain space:

(i) if $\varphi_{\beta,\beta}$ is a coboundary in the space of bounded functions, then $\beta \in \mathbb{Z}\alpha + \mathbb{Z}$ (cf. Shapiro's result);

(ii) if $\sum b_k^4/a_{k+1} < \infty$, then $\varphi_{\beta,\beta}$ is a coboundary with a transfer function in L^2 (see Theorem 2.2);

(iii) if $\sum |b_k|/a_{k+1} < \infty$, then $\varphi_{\beta,\beta}$ is a coboundary with a measurable transfer function. (Proposition 3.2). This is also necessary by Theorem 2.3 b).

3.2.2. *Example of trivial centralizer.*

Now, for α of bounded type, we show the triviality of the centralizer in the special case $\beta = \frac{1}{2}$.

Theorem 3.4. *Let α be of bounded type. For $\beta = \frac{1}{2}$, the centralizer of T_{α,φ_β} (acting on $X \times \frac{1}{2}\mathbb{Z}$) reduces to the translations on the fibers $(x, y) \to (x, y+\lambda)$, for a constant $\lambda \in \mathbb{R}$, the map $(x, y) \to (x + \frac{1}{2}, -y)$ and the powers of T_{α,φ_β}.*

Proof. The cocycle $\varphi = \varphi_{\frac{1}{2},\frac{1}{2}} = 2\varphi_{\frac{1}{2}}$ is known to be ergodic as a cocycle with values in $\mathbb{Z}$, for every irrational rotation ([6]).

According to Theorem 1.4 and the commutation relation (6), we consider the cocycle $u_\gamma := \varepsilon\varphi - T_\gamma\varphi$, where ε is the constant $+1$ or -1. Suppose that α is of bounded type and $\gamma \notin \mathbb{Z}\alpha + \mathbb{Z}$.

Assume that $\gamma \neq \frac{1}{2}$ mod 1, so that u_γ has effective discontinuities for $x = 0, \frac{1}{2}, -\gamma, \frac{1}{2} - \gamma$.

By Lemma 2.3 and Theorem 3.8 in [9] to which we refer for more details, the cocycle u_γ satisfies a property of separation of its discontinuities along a subsequence of denominators of α and therefore its discontinuities belong to the group of its finite essential values.

This implies that u_γ has a non trivial essential value, hence is not a coboundary.

The case $\gamma = \frac{1}{2}$ mod 1 corresponds to the special map $(x, y) \to (x + \frac{1}{2}, -y)$ which yields an element in the centralizer due to the relation satisfied here: $-\varphi(x) = \varphi(x + \frac{1}{2})$.

It remains to examine the case $\gamma = p\alpha \mod 1$, with $p \neq 0$ in $\mathbb{Z}$.

Suppose that $\varepsilon = -1$. Then $\varphi + T_{p\alpha}\varphi$ is a T_α-coboundary, hence also φ, since $\varphi - T_{p\alpha}\varphi = (\varphi + ... + T_{(p-1)\alpha}\varphi) - T_\alpha(\varphi + ... + T_{(p-1)\alpha}\varphi)$ is a coboundary.

Since φ is not a coboundary, necessarily $\varepsilon = +1$.

For $\varepsilon = +1$ and $\gamma = p\alpha + \ell$, we find the powers of the map T_{α,φ_β}. □

3.3. **Example of a non trivial conjugacy in a group family.**

Another application is a conjugacy problem for a family of closed subgroup over a dynamical system.

We consider the following data: a dynamical system (X, μ, T), a measurable family $(H_x)_{x\in X}$ of closed subgroups of a (non commutative) topological group G and a measurable function $\Phi : X \to G$ such that the following conjugacy equation holds:

$$H_{Tx} = \Phi(x)\ H_x\ (\Phi(x))^{-1}, \text{ for } \mu_\chi\text{–a.e. } x \in X. \tag{24}$$

We would like to give a simple example of construction of such a family which is *not conjugate to a fixed closed subgroup* of G (cf. [8]), i.e., such that there is no subgroup $H \subset G$ and no measurable function $\zeta : X \to G$ solution of the equation

$$H_x = \zeta(x)^{-1} H \zeta(x). \tag{25}$$

Let θ be a fixed irrational number and let G be the solvable group obtained as the semi-direct product of $\mathbb{R}$ and $\mathbb{C}^2$, with the composition law:

$$(t, z_1, z_2) * (t', z_1', z_2') = (t + t', z_1 + e^{2\pi it} z_1', z_2 + e^{2\pi\theta it} z_2').$$

The conjugate of $(0, z_1, z_2)$ by $a = (s, v_1, v_2)$ in G is:

$$(s, v_1, v_2)(0, z_1, z_2)(s, v_1, v_2)^{-1} = (0, e^{2\pi i s} z_1, e^{2\pi\theta i s} z_2). \tag{26}$$

Consider the dynamical system defined by an irrational rotation $T : x \to x+\alpha \mod 1$ on $X = \mathbb{T}^1$. Let $\Phi : X \to G$ be the cocycle defined by $\Phi(x) = (\varphi(x), 0, 0)$, where φ has its values in $\mathbb{Z}$.

Let $H_x := \{(0, v z_1, v e^{2\pi i \psi(x)} z_2), v \in \mathbb{R}\}$, where ψ is a measurable real function defined below and z_1, z_2 are given real numbers. For every $x \in X$, H_x is a closed subgroup of G. Let us consider the function $x \to H_x$ with values in the set of closed subgroups of G. It satisfies the conjugacy relation (24) if and only if φ has integral values and satisfies

$$\theta\ \varphi(x) + \psi(x) = \psi(Tx) \mod 1. \tag{27}$$

Let us take $\varphi = \varphi_{\beta,\gamma} = 1_{[0,\beta]} - 1_{[0,\beta]}(. + \gamma)$. We have seen that, for every α which is not of bounded type, there are real numbers β and γ for which the function $\varphi_{\beta,\gamma}$ is not a coboundary and $e^{2\pi i \theta \varphi_{\beta,\gamma}}$ is a multiplicative coboundary for some irrational values of θ.

It means that for these values of the parameters, there is ψ such that (27) is satisfied

Proposition 3.5. *For these choices of β, θ, $\varphi = \varphi_{\beta,\gamma}$ and ψ, there is no subgroup H such that the equation (25) has a measurable solution ζ.*

Proof. Suppose that there are a fixed subgroup H and a measurable function $\zeta : X \to G$ solution of (25). According to (26), this is equivalent to the existence of a function ρ defined on X such that the set

$$\{(0, v e^{2\pi i \rho(x)} z_1, v e^{2\pi i(\theta\rho(x) + \psi(x))} z_2), v \in \mathbb{R}\}$$

does not depend on x. This implies that ρ and $\psi + \theta\rho$ have a fixed value mod 1. Therefore $\rho(x) - \rho(Tx) \in \mathbb{Z}$, $\theta(\varphi(x) - \rho(x) + \rho(Tx)) = \theta\varphi(x) + \psi(x) - \psi(Tx)$ and according to (27) the common value mod 1 is 0.

As φ has integral values and θ is irrational, it follows that $\varphi = T\rho - \rho$, contrary to the fact that φ is not a coboundary. □

4. **Appendix: proof of Theorem 2.2**

For the proof of Theorem 2.2 we need some preliminary results. In what follows, C will denote a generic constant which may change from a line to the other.

Bounds for $\|q_n\beta\|$

Let $\beta \in [0, 1]$ be such that

$$\beta = \sum_1^\infty b_i q_i \alpha \mod 1, \text{ with } \sum_1^\infty \frac{|b_i|}{a_{i+1}} = C_1 < \infty. \tag{28}$$

In the following computations, we assume that there is infinitely many i's with $b_i \neq 0$. We can assume $b_i \geq 0$.

The quantities $\|q_n\beta\|$ and $|b_n|/a_{n+1}$ are of the same order. For all $r \geq 1$ such that $b_r \neq 0$, the following upper bounds hold:

$$\sum_{j=1}^{r} b_j q_j \leq q_r(b_r + \frac{b_{r-1}}{a_r} + \frac{b_{r-2}}{a_r a_{r-1}} + ... + \frac{b_1}{a_r a_{r-1} ... a_2}) \leq q_r(b_r + C_1) \leq (C_1+1) b_r q_r,$$

$$\sum_{j=r}^{\infty} b_j \|q_j\alpha\| \leq \frac{b_r}{q_{r+1}} + \frac{b_{r+1}}{q_{r+2}} + ... \leq \frac{1}{q_{r+1}}(b_r + \frac{b_{r+1}}{a_{r+2}} + \frac{b_{r+2}}{a_{r+2}a_{r+3}} + ...)$$

$$\leq \frac{1}{q_{r+1}}(b_r + C_1) \leq (C_1+1)\frac{b_r}{q_{r+1}}.$$

For $n \geq 1$, let $\ell(n)$ be the greatest index $i \leq n-1$ such that $b_i \neq 0$, and $m(n)$ the smallest index $i \geq n$ such that $b_i \neq 0$. For all $r, k \geq 1$, we have

$$\|k\beta\| = \|\sum_{1}^{\infty} b_i q_i k\alpha\| \leq \min(1,\, 2\max(\|k\alpha\| \sum_{1}^{r-1} b_i q_i,\, k \sum_{r}^{\infty} b_i \|q_i\alpha\|));$$

hence with $C = 2(C_1+1)$:

$$\|k\beta\| \leq \min(1,\, C\max(\|k\alpha\|\, b_{\ell(n)} q_{\ell(n)},\, k\frac{b_{m(n)}}{q_{m(n)+1}})),\ \forall n, k \geq 1. \tag{29}$$

Observe that since $b_{\ell(n)}$ is non zero integer,

$$\sum_n \frac{1}{a_{\ell(n)+1}} \leq \sum_n \frac{|b_{\ell(n)}|}{a_{\ell(n)+1}} < C_1. \tag{30}$$

We will use also that if s is an integer ≥ 1, then

$$\sum_j \frac{b_j^s}{a_{j+1}} < +\infty \Rightarrow \sum_n \frac{b_{\ell(n)}^s q_{\ell(n)}}{q_{\ell(n)+1}} < +\infty.$$

Denjoy-Koksma inequality (cf. [13])

We denote by $V(f)$ the variation of a BV (bounded variation) function f on $X = \mathbb{R}/\mathbb{Z}$, for instance a step function with a finite number of discontinuities. If p/q is a irreducible fraction such that $\|\alpha - p/q\| < 1/q^2$, then for every $x \in X$ the following inequality holds:

$$|\sum_{\ell=0}^{q-1} f(x+\ell\alpha) - q\int f\, dy| \leq V(f). \tag{31}$$

Let $S_n f = \sum_{k=0}^{n-1} T_\alpha^k f$ be the Birkhoff sums of f for the rotation T_α. Using Inequality (31) implies for the denominators q_n of α:

$$\|S_{q_n} f\|_\infty \leq |\mu(f)|\, q_n + V(f), \forall n \in \mathbb{N}. \tag{32}$$

Lemma 4.1. *If f is a nonnegative BV function, we have,*

$$\sum_{k=q_n}^{\infty} \frac{f(k\alpha)}{k^2} \leq 2(\frac{\mu(f)}{q_n} + \frac{V(f)}{q_n^2}), \ \forall n \geq 1. \tag{33}$$

Proof. The inequality (32) implies

$$\begin{aligned} \sum_{k=q_n}^{\infty} \frac{f(k\alpha)}{k^2} &\leq \sum_{j=1}^{\infty} \frac{1}{(jq_n)^2} \sum_{p=0}^{q_n-1} f((jq_n+p)\alpha) \\ &\leq \frac{1}{q_n^2} (\sum_{j=1}^{\infty} \frac{1}{j^2}) \, (\mu(f)\, q_n + V(f)) = 2(\frac{\mu(f)}{q_n} + \frac{V(f)}{q_n^2}). \end{aligned}$$

□

For all $p \geq 1$, by (33) applied with $f(x) = \frac{1}{x^2} 1_{[\frac{1}{p}, \frac{1}{2}]}(|x|)$, then applied with $f(x) = 1_{[-\frac{1}{p}, \frac{1}{p}]}(x)$, we get

$$\sum_{\{k \geq q_n, \, \|k\alpha\| \geq 1/p\}} \frac{1}{k^2} \frac{1}{\|k\alpha\|^2} \leq C(\frac{p}{q_n} + \frac{p^2}{q_n^2}), \tag{34}$$

$$\sum_{\{k \geq q_n, \, \|k\alpha\| \leq 1/p\}} \frac{1}{k^2} \leq C(\frac{1}{q_n p} + \frac{1}{q_n^2}). \tag{35}$$

On the other hand, we have, from (31):

$$\begin{aligned} \sum_{\{0<k<q_n, \, \|k\alpha\| \geq 1/p\}} \frac{1}{\|k\alpha\|^2} &= \sum_{\{0<k<q_n\}} \frac{1}{\|k\alpha\|^2} \, 1_{[1/p, \, 1-1/p]}(\{k\alpha\}) \\ &\leq 2p^2 \sum_{\ell=1}^{[\frac{p+1}{2}]} \frac{1}{\ell^2} \sum_{\{0<k<q_n\}} 1_{[\frac{\ell}{p}, \frac{\ell+1}{p}[}(\{k\alpha\}) \ \leq 2p^2(\frac{q_n}{p} + 2) \sum_{\ell \geq 1} \frac{1}{\ell^2}; \end{aligned}$$

hence:

$$\sum_{\{0<k<q_n, \, \|k\alpha\| \geq 1/p\}} \frac{1}{\|k\alpha\|^2} \leq Cp(q_n + p). \tag{36}$$

Lemma 4.2. *a) There is a finite constant C such that, for every $n \geq 1$,*

$$\sum_{k=1}^{q_n-1} \frac{1}{\|k\alpha\|^2} \leq Cq_n^2. \tag{37}$$

b) For all $s \geq 1$, there exists at most one value of k of the form $k = sq_n + r$, with $r \in [1, q_n[$ such that $\|k\alpha\| < \frac{1}{4}\frac{1}{q_n}$, and this value satisfies $k \geq \frac{1}{4}q_{n+1}$.

Proof. a) If $\alpha > \frac{p_n}{q_n}$, then each interval $[\frac{j}{q_n}, \frac{j+1}{q_n})$, $1 \leq j \leq q_n - 1$ contains exactly one number of the form $\{k\alpha\}$, with $1 \leq k \leq q_n - 1$. Therefore we have

$$\sum_{k=1}^{q_n-1} \frac{1}{\|k\alpha\|^2} \leq \sum_{\ell=1}^{q_n-1} (\ell/q_n)^{-2} \leq q_n^2 \sum_{\ell=1}^{+\infty} \ell^{-2}. \tag{38}$$

When $\alpha < \frac{p_n}{q_n}$, the same is true for $j = 1, \dots, q_n - 2$. Furthermore there is an exceptional value k_1 (the value such that $k_1 p_n = 1 \mod q_n$) for which $0 < \{k_1\alpha\} < \frac{1}{q_n}$. By (13) we know that $\|k\alpha\| \geq \frac{1}{2q_n}$ for $1 \leq |k| < q_n$. Therefore $\frac{1}{2q_n} < \{k_1\alpha\} < \frac{1}{q_n}$ which add a contribution of $4q_n^2$ in (38). This implies (38).

b) For a given integer $s \geq 1$, suppose that there are two different values of the form $k_i = sq_n + r_i$, with $r_i \in [1, q_k[$, $i = 1, 2$, satisfying: $\|k_i\alpha\| < \frac{1}{4}\frac{1}{q_n}$.

Then we have, for some $r_0 \in [1, q_n[$, $\|r_0\alpha\| < \frac{1}{2}\frac{1}{q_n}$, which contradicts that for $r_0 \in [1, q_n[$ we have $\|r_0\alpha\| \geq \|q_{n-1}\alpha\| \geq \frac{1}{2q_n}$ by (12) and (13).

Let $k = sq_n + r$, with $r \in [1, q_n[$ and $\|k\alpha\| < \frac{1}{4}\frac{1}{q_n}$. Put $\lambda = sq_n/q_{n+1}$. The condition $\|k\alpha\| < \frac{1}{4}\frac{1}{q_n}$ implies

$$\|r\alpha\| < \frac{1}{4q_n} + \frac{s}{q_{n+1}} \leq (\lambda + \frac{1}{4})\frac{1}{q_n}.$$

As $\|r\alpha\| \geq \|q_{n-1}\alpha\| \geq \frac{1}{2q_n}$, for $r \in [1, q_n[$, we get $\lambda > \frac{1}{4}$. □

The proof of Theorem 2.2 relies on the expansion of β in basis $q_n\alpha$ mod 1. We suppose that $\beta \in [0, 1[$ satisfies (28), so that we can apply (29).

We denote by J and J' the sets of integers defined by

$$J := \{k = sq_n, s = 1, ..., a_{n+1}, n = 1, 2, ...\}, \tag{39}$$

$$J' := \bigcup_{n=1}^{\infty} ([q_n, q_{n+1}[\cap \{k : \|k\alpha\| < \frac{1}{4q_n}\}). \tag{40}$$

Lemma 4.3.

$$\sum \frac{b_j(\beta)^2}{a_{j+1}} < \infty \ \Rightarrow \sum_{n \neq 0, n \notin J} \frac{1}{n^2} \frac{\|n\beta\|^2}{\|n\alpha\|^2} < \infty. \tag{41}$$

Proof. Up to a constant factor, we have

$$\begin{aligned}
&\sum_{k\neq 0, k\notin J} \frac{\|k\beta\|^2}{k^2\|k\alpha\|^2} = \sum_{n=0}^{\infty} \sum_{k\notin J,\, q_n\leq k<q_{n+1}} \frac{\|k\beta\|^2}{k^2\|k\alpha\|^2} \\
&\leq \sum_{n=0}^{\infty} \sum_{q_n\leq k<q_{n+1},\, \|k\alpha\|\geq 1/q_{\ell(n)}} \frac{\|k\beta\|^2}{k^2\|k\alpha\|^2} + \sum_{n=0}^{\infty} \sum_{k\notin J,\, q_n\leq k<q_{n+1},\, \|k\alpha\|<1/q_{\ell(n)}} \frac{\|k\beta\|^2}{k^2\|k\alpha\|^2} \\
&\leq \sum_{n=0}^{\infty} \sum_{q_n\leq k<q_{n+1},\, \|k\alpha\|\geq 1/q_{\ell(n)}} \frac{1}{k^2\|k\alpha\|^2} + \sum_{n=0}^{\infty} \sum_{k\notin J,\, q_n\leq k<q_{n+1},\, \|k\alpha\|<1/q_{\ell(n)}} \frac{\|k\beta\|^2}{k^2\|k\alpha\|^2} \\
&\leq (A) + (B) + (C) + (D),
\end{aligned}$$

with (using (29) for (B) and (C)):

$$\begin{aligned}
(A) &:= \sum_n \sum_{q_n\leq k<q_{n+1},\, \|k\alpha\|\geq 1/q_{\ell(n)}} \frac{1}{k^2\|k\alpha\|^2}, \\
(B) &:= \sum_n \sum_{q_n\leq k<q_{n+1},\, \|k\alpha\|<1/q_{\ell(n)}} b_{\ell(n)}^2 q_{\ell(n)}^2 \frac{1}{k^2}, \\
(C) &:= \sum_n \sum_{k\notin J\cup J',\, q_n\leq k<q_{n+1}} b_{m(n)}^2 \frac{1}{q_{m(n)+1}^2} \frac{1}{\|k\alpha\|^2}. \\
(D) &:= \sum_{k\notin J,\, k\in J'} \frac{\|k\beta\|^2}{k^2\|k\alpha\|^2}.
\end{aligned}$$

Observe that $q_n \geq q_{\ell(n)+1} > a_{\ell(n)+1}\, q_{\ell(n)}$, since $\ell(n)+1 \leq n$. We have from (34) and from (35) applied with $p = q_{\ell(n)}$ and from (30):

$$\begin{aligned}
(A) \quad &\leq C\sum_n \Big(\frac{q_{\ell(n)}}{q_n} + \frac{q_{\ell(n)}^2}{q_n^2}\Big) \leq C\sum_j \Big(\frac{1}{a_{\ell(n)+1}} + \frac{1}{a_{\ell(n)+1}^2}\Big) < \infty, \\
(B) \quad &\leq C\sum_n b_{\ell(n)}^2 q_{\ell(n)}^2 \Big(\frac{1}{q_n q_{\ell(n)}} + \frac{1}{q_n^2}\Big) \\
&\leq C\sum_n b_{\ell(n)}^2 \Big(\frac{q_{\ell(n)}}{q_n} + \frac{q_{\ell(n)}^2}{q_n^2}\Big) \leq \sum_j \frac{b_j^2}{a_{j+1}}\Big(1 + \frac{1}{a_{j+1}}\Big) < +\infty,
\end{aligned}$$

and from (36), as the sum is taken over indices $k \notin J'$, i.e. such that $\|k\alpha\| \geq \frac{1}{4q_n}$:

$$\begin{aligned}
(C) \quad &\leq C\sum_n b_{m(n)}^2 \frac{4q_n(q_{n+1}+4q_n)}{q_{m(n)+1}^2} \\
&\leq C\sum_n b_{m(n)}^2 \frac{q_n}{q_{m(n)+1}} \leq C'\sum_n \frac{b_{m(n)}^2}{a_{m(n)+1}} \leq C'\sum_j \frac{b_j^2}{a_{j+1}} < +\infty.
\end{aligned}$$

We are left with the convergence of the series (D). By (29) it suffices to prove the following convergence

$$(E) := \sum_n \sum_{k \notin J, k \in J',\, q_n \leq k < q_{n+1}} b^2_{\ell(n+1)} q^2_{\ell(n+1)} \frac{1}{k^2} < \infty,$$

$$(F) := \sum_n \sum_{k \notin J, k \in J',\, q_n \leq k < q_{n+1}} b^2_{m(n+1)} \frac{1}{q^2_{m(n+1)+1}} \frac{1}{\|k\alpha\|^2} < \infty.$$

By Lemma 4.2, we have:

$$(E) \leq C \sum_n b^2_{\ell(n+1)}\, q^2_{\ell(n+1)} \frac{a_{n+1}}{q^2_{n+1}} \leq C \sum_n b^2_{\ell(n+1)} \frac{q_{\ell(n+1)}}{q_{\ell(n+1)+1}} < \infty.$$

To bound (F), we use Lemma 4.2 a):

$$\begin{aligned} (F) \qquad &\leq C \sum_n b^2_{m(n+1)} \frac{1}{q^2_{m(n+1)+1}} q^2_{n+1} (\sum_{\ell=1}^{\infty} \frac{1}{\ell^2}) \\ &\leq C \sum_n b^2_{m(n+1)} \frac{q^2_{m(n+1)}}{q^2_{m(n+1)+1}} (\sum_{\ell=1}^{\infty} \frac{1}{\ell^2}) \leq C \sum \frac{b_j^2}{a^2_{j+1}} < \infty. \end{aligned}$$

$\square$

Proof of Theorem 2.2 Let $\beta \in H_r(\alpha)$, i.e.,

$$\sum \frac{b_j(\beta)^4}{a_{j+1}} < \infty, \tag{42}$$

Taking into account Lemma 4.3, it remains to show the convergence of

$$\sum_{n \neq 0} \sum_{s=1}^{a_{n+1}} \frac{\|s q_n \beta\|^4}{s^2 q_n^2 \|s q_n \alpha\|^2}.$$

By (29) applied with $k = q_n$, it suffices to prove the convergence of the series

$$(G) := \sum_n b^2_{\ell(n)} q^2_{\ell(n)} \frac{1}{q_n^2} (\sum_{s=1}^{a_{n+1}} \frac{1}{s^2}),$$

$$(H) := \sum_n b^4_{m(n)} \frac{1}{q^4_{m(n)+1}} (\sum_{s=1}^{a_{n+1}} q_n^2 q^2_{n+1}).$$

Since $n \leq m(n)$ and $\ell(n) + 1 \leq n$, we have from (42):

$$(G) \qquad \leq C \sum_n b^2_{\ell(n)} \frac{q^2_{\ell(n)}}{q_n^2} \leq C \sum_n b^2_{\ell(n)} \frac{q^2_{\ell(n)}}{q^2_{\ell(n)+1}} \leq C \sum_n \frac{b^2_{\ell(n)}}{a^2_{\ell(n)+1}} \leq \sum_j \frac{b_j^2}{a^2_{j+1}} \leq \sum_j \frac{b_j^2}{a_{j+1}} < \infty,$$

$$\begin{aligned} (H) \qquad &\leq C \sum_n b^4_{m(n)} \frac{a_{n+1} q_n^2 q^2_{n+1}}{q^4_{m(n)+1}} \leq C \sum_n b^4_{m(n)} \frac{q_n q^3_{n+1}}{q^4_{m(n)+1}} \leq C \sum_n b^4_{m(n)} \frac{q_{m(n)} q^3_{m(n)+1}}{q^4_{m(n)+1}} \\ &= C \sum_n b^4_{m(n)} \frac{q_m(n)}{q_{m(n)+1}} \leq C \sum_n \frac{b^4_{m(n)}}{a_{m(n)+1}} \leq \sum_j \frac{b_j^4}{a_{j+1}} < \infty. \end{aligned}$$

Therefore, if $\beta \in H_4(\alpha)$, we have $\sum_{n\neq 0} \frac{1}{n^2} \frac{\|n\beta\|^4}{\|n\alpha\|^2} < \infty$.

For the second statement of Theorem 2.2, observe that, if β and γ belong to $H_4(\alpha)$, by the previous inequality and Cauchy-Schwarz inequality

$$\sum_{n\neq 0} \frac{1}{n^2} \frac{\|n\beta\|^2 \|n\gamma\|^2}{\|n\alpha\|^2} < \infty. \tag{43}$$

Recall that the Fourier coefficients of $\varphi_{\beta,\gamma}$ are $\frac{1}{2\pi i n}(e^{2\pi i n\beta} - 1)(e^{2\pi i n\gamma} - 1)$. The condition for $\varphi_{\beta,\gamma}$ to be a coboundary with a transfer function in $L^2(\mathbb{T}^1)$, i.e., such that the functional equation $\varphi_{\beta,\gamma} = T_\alpha h - h$ has a solution h in L^2, is fulfilled by (43). □

References

[1] J. Aaronson. *An Introduction to Infinite Ergodic Theory*, Mathematical Surveys and Monographs **50**, AMS, Providence, 1997.

[2] J. Aaronson, M. Lemańczyk, C. Mauduit, H. Nakada. *Koksma's inequality and group extensions of Kronecker transformations.* Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992), 27-50, Plenum, New York, 1995.

[3] J. Aaronson, M. Lemańczyk, D. Volný. *A cut salad of cocycles. Dedicated to the memory of Wieslaw Szlenk.* Fund. Math. 157 (1998), no. 2-3, 99-119.

[4] V. I. Arnold. *Small denominators I. Mapping the circle onto itself.* (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961) 21-86.

[5] J.-P. Conze. *Equirépartition et ergodicité de transformations cylindriques*, Publications des Séminaires de Rennes, Fasc. I Proba., Rennes (1976).

[6] J.-P. Conze, M. Keane. *Ergodicité d'un flot cylindrique*, Publications des Séminaires de Rennes, Fasc. I Proba., Rennes (1976).

[7] J.-P. Conze. *Recurrence, ergodicity and invariant measures for cocycles over a rotation*, Contemporary Mathematics, vol. 485 (2009), p. 45-70.

[8] J.-P. Conze, A. Raugi. *On the ergodic decomposition for a cocycle*, Colloquium Mathematicum, vol. 117, 2009 No.1.

[9] J.-P. Conze, A. Piękniewska. *On multiple ergodicity of affine cocycles over irrational rotations*, preprint 2012. arXiv:1209.3798

[10] K. Frączek, C. Ulcigrai. *Non-ergodic $\mathbb{Z}$-periodic billiard and infinite translation surfaces*, preprint 2011, arXiv:1109.4584.

[11] M. Guénais, F. Parreau. *Valeurs propres de transformations liées aux rotations irrationnelles et aux fonctions en escaliers*, preprint 2006, arXiv:math/0605250

[12] G.H. Hardy, E.M. Wright. *An introduction to the theory of numbers*, Oxford science publications.

[13] M.-R. Herman. *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, (French) Inst. Hautes Études Sci. Publ. Math. No. 49 (1979), 5-233.

[14] O. Jenkinson. *Strong cocycle triviality for $\mathbb{Z}^2$-subshifts*, Theoretical Computer Science 262 (2001), p. 191-213.

[15] J. Kwiatkowski. *Factors of ergodic group extensions of rotations*, Studia Math. 103 (1992), no. 2, p. 123-131.

[16] M. Lemańczyk. *Cohomology groups, multipliers and factors in ergodic theory*, Studia Math. 122 (1997), no. 3, 275-288.

[17] M. Lemańczyk, M. Mentzen, H. Nakada. *Semisimple extensions of irrational rotations*, Studia Math. **156** (2003).

[18] M. Lemanczyk, F. Parreau, D. Volný. *Ergodic properties of real cocycles and pseudo-homogeneous Banach spaces*, Trans. Amer. Math. Soc. 348 (1996), no. 12, 4919-4938.

[19] E. Lesigne. *Equations fonctionnelles, couplages de produits gauches et théorèmes ergodiques pour mesures diagonales*, S.M.F., t. 121, no 3 (1993), p. 315-351.

[20] K. Merrill. *Cohomologie of step functions under irrational rotations*, Israel J. Math. 52 (1985), no. 4, p. 320-340.

[21] C.C. Moore, K. Schmidt. *Coboundaries and homomorphisms for nonsingular actions and a problem of H. Helson*, Proc. London Math. Soc. (3) 40 (1980), no. 3, 443-475.

[22] K. Petersen. *On a series of cosecants related to a problem in ergodic theory*, Compositio Math. 26 (1973), 313-317.

[23] I. Oren. *Ergodicity of cylinder flows arising from irregularities of distribution*, Israel J. Math. 44 (1983), no. 2, p. 127-138.

[24] K. Schmidt. *Lectures on cocycles of ergodic transformation groups*, Macmillan Lectures in Mathematics, Vol. 1., Macmillan Company of India, Ltd., Delhi, 1977, 202 pp.

[25] K. Schmidt. *The cohomology of higher-dimensional shifts of finite type*, Pacific Journal of Mathematics, Vol. 170 No. 1 (1995) p. 237-269,

[26] L. Shapiro. *Irregularities of distribution in dynamical systems.* Recent advances in topological dynamics (Proc. Conf. Topol. Dyn., Yale Univ., New Haven, Conn., 1972, in honor of G. A. Hedlund), pp. 249-252. Lecture Notes in Math., Vol. 318, Springer, Berlin, 1973.

[27] W. A. Veech. *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2*, TAMS Vol. 140 (1969), p. 1-33.

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